Problem Solving Session: Number Theory
Department of Mathematics, Andrews University

Strategy: Eat pancakes! Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Discuss. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont give up after five minutes. Come back to it later. Ask.

1. Show $\sqrt{6}$ is irrational.

2. Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$. (Putnam 1955/A1)

3. Let n be a positive integer. Prove that $n(n+1)(n+2)(n+3)$ cannot be a square or a cube. (Putnam 1959/B2)

4. Show that when you multiply an integer by 9 (or a multiple of 9), the digits of the resulting number sum to a multiple of 9.

5. Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10. (Putnam 1956/A2)

6. Show that for any positive integer r, we can find integers m, n such that $m^2 - n^2 = r^3$.

7. Show that the sum of consecutive primes is never twice a prime.

8. Show that there exists an increasing sequence $(a_n)_{n\geq 1}$ of positive integers os that for any $k \geq 0$ the sequence $(k + a_n)_{n\geq 1}$ contains only finitely many primes. (Putnam and Beyond #789)

9. Show 13 divides $2^{70} + 3^{70}$. Use *Fermat's Little Theorem*: for prime p and integer a,

$$a^p \equiv a \pmod{p}.$$