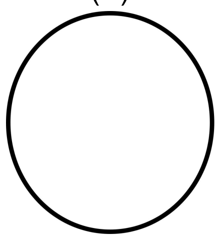
Reideimesiter Moves

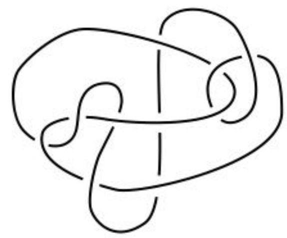
A knot is an embedding of a circle into three dimensional space. We say that two knots are equivalent if one can be deformed into the other without it passing through itself. More precisely, we say that two knots K_1 and K_2 are equivalent if they are *ambient isotopic*. A central question in knot theory is how to tell if two knots are equivalent or not.

Take some time to convince youself that knots (A) and (B) below are equivalent--these are called *unknots* because they are not knotted. Perhaps less obvious, though, is if (C) is equivalent to (A). How can we know for sure that there is no really clever way to deform (C) into (A)?

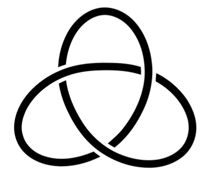
(A)



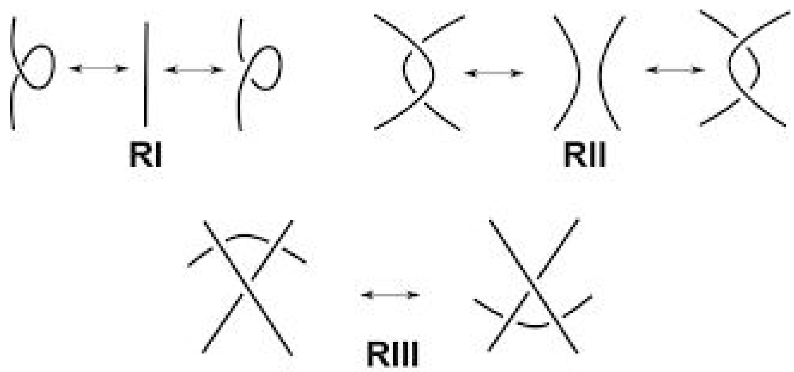
(B)



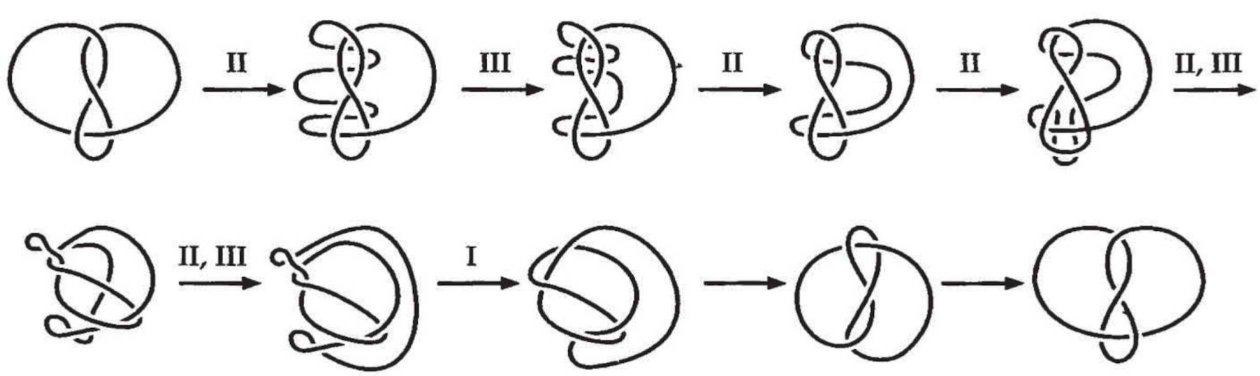
(C)



A Reideimesiter move is one of three ways (I. twist, II. poke, III. slide) to change a knot diagram into an equivalent knot. They are as folllows:



It's obvious that if we can perform a Reideimesiter move--or sequence of such moves--that we end up with an equivalent knot. Much more surprising is that any two diagrams of equivalent knots can transformed from one to the other via a sequence of Reideimestier moves. For instance, here's the sequence of moves transforming the Figure Eight knot into its mirror image:

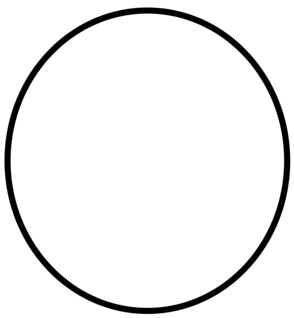
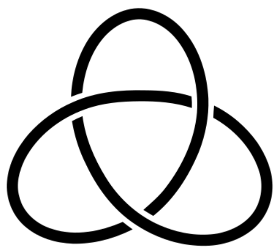
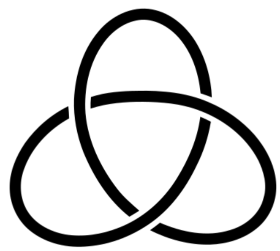
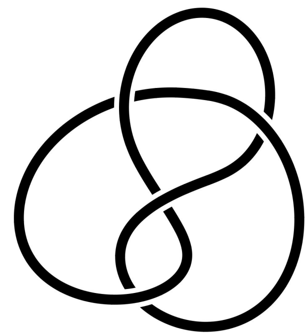
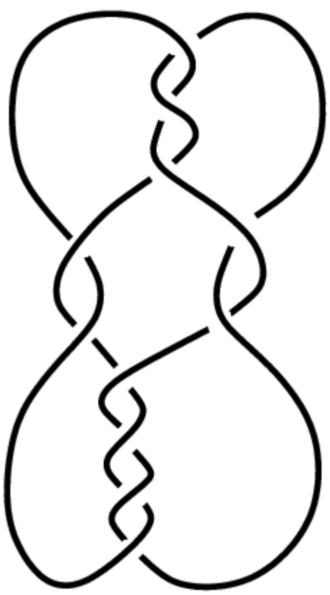
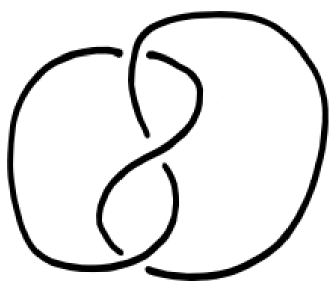
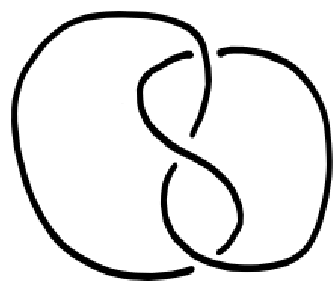
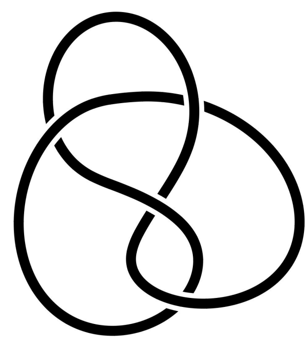


However, it is not computationally feasible to use Reideimeister moves to check if two knots are equivalent. For instance, Lackendy (2015) proved that if a knot with c crossings is equivalent to the unknot, then an upperbound on the number of Reideimeister moves required to move the diagram into the standard unknot diagram is $(236c)^{11}$.

Colorings

Definition: A knot diagram is called <u>colorable</u> if each arc can be drawn using one of three colors such that
1) at least two of the colors are used and
2) at any crossing at which two colors appear, all three appear. That is, at every crossing, either all the colors are the same or they are all different.

Determine which of the below diagrams are colorable:

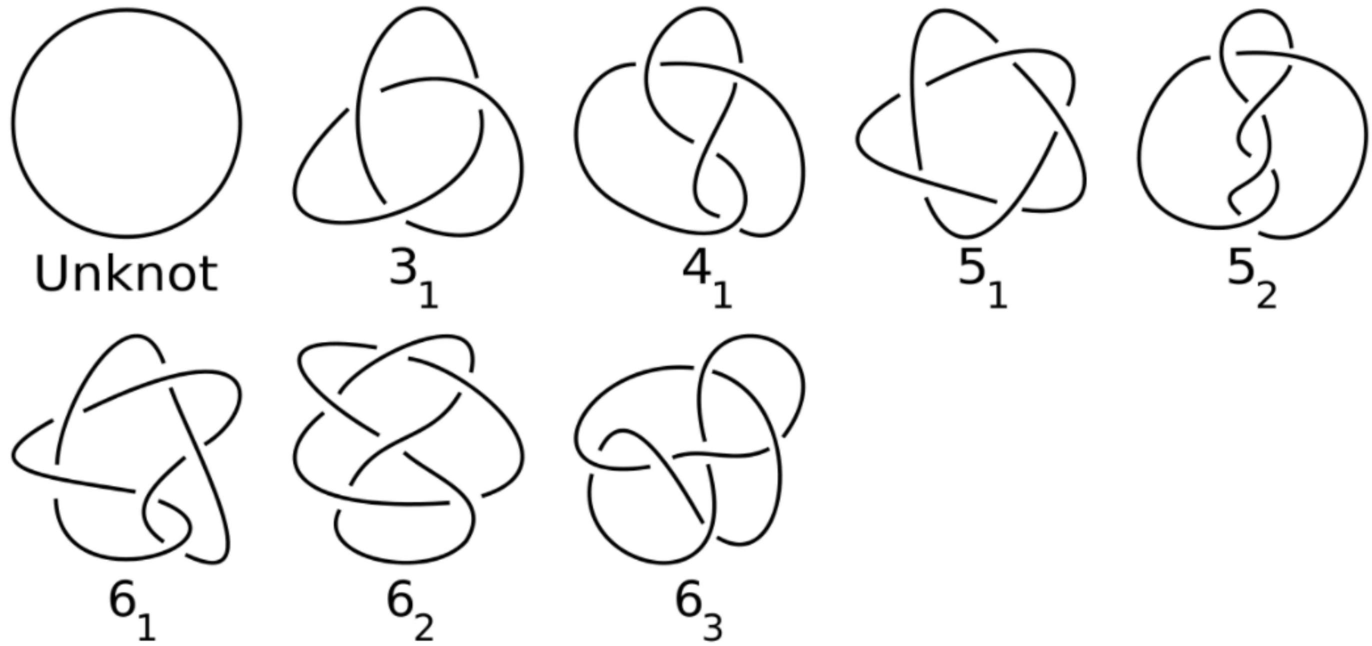
Unknot	Right-handed Trefoil	Left-handed Trefoil	Figure Eight
			
			

You may have noticed something from the above example:
Theorem: If a diagram of a knot K is colorable, then every diagram of K is colorable.

Problem: Prove this by showing that each of the three Reideimester moves preserves colorability. That is, show that if a knot diagram is colorable if and only if it is still colorable after performing any of the Reideimester moves. Since any two equivalent knots are related by a series of Reideimester moves, this proves the theorem.

We can therefore offer the following definition.
Definition: A knot is called <u>colorable</u> if any (and hence all) of its diagrams are colorable.

Hence, colorablity distinishes knots into two types--thosse that are colorable and those that are not. Determine which of the below knots are colorable. Notice, we still cannot distinguish, though, if two colorable knots are equivalent to each other or not.



Notice, *Condition (2)* above can be expressed in terms of modular arithmetic as follows: If at a crossing the undercrossings are labeled x and y and the overcrossing is labeled z, then $x + y \equiv 2z \ (mod\ 3)$. We can nownaturally generalize colorability to p-colorability for any prime $p \geq 3$ as follows:

Definition: A knot is called p*-colorable* if in one (and hence all) of its diagrams each arc can be labeled with a value from $\{0, ..., p-1\}$ such that:

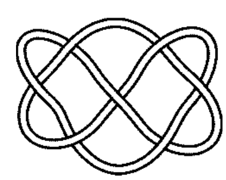
1) at least two distinct labels are used and
2) if at a crossing the undercrossings are labeled x and y and the overcrossing is labeled z, then $x + y \equiv 2z \ (mod\ p)$.

Problem: Show that no knot is 2-colorable. To do this, think about what would happen in the above definition if $p = 2$.

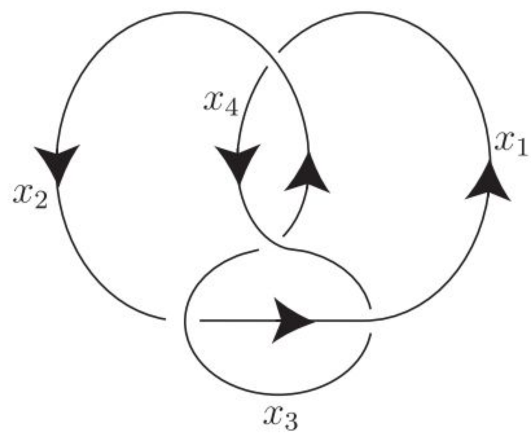
Matrices

Theorem: There is an $n \times n$ matrix corresponding to a knot diagram K with n arcs. Delete any one column and any one row to obtain the matrix M_K. The knot is colorable if and only if the determinant of the matrix is divisible by 3. More generally, the knot is p-colorable if and only if the determinant of the matrix is divisible by p.

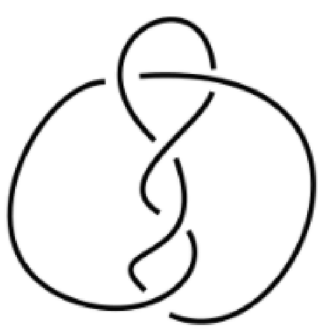
Example: To the right is the knot 7_4. It has $det(7_4) = 15$, therefore the knot is 3-colorable and 5-colorable, but not, say, 7-colorable or 11-colorable.
Can you find an example of a 3-labeling and 5-labeling?



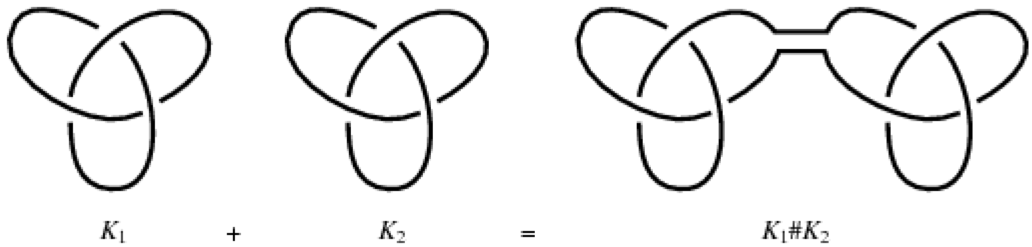
Example: What is the matrix of the figure eight knot pictured below?



Problem: Calculate the determinant of the below knot. For what primes p is it p-colorable?



We now introduce a way of combining two knots to get a new one. This operation is called connect sum.



Consider the following questions. Try some examples to guess the answer.
If both K_1 and K_2 are mod p colorable, then is the connected sum $K_1\#K_2$ mod p colorable?
Sometimes? Always? Never?

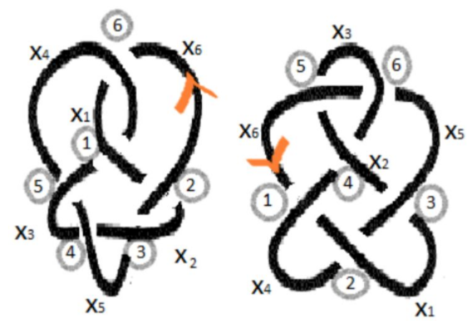
If only one of K_1 and K_2 are mod p colorable, then is the connected sum $K_1\#K_2$ mod p colorable?
Sometimes? Always? Never?

If neither of K_1 and K_2 are mod p colorable, then is the connected sum $K_1\#K_2$ mod p colorable?
Sometimes? Always? Never?

Our goal is to answer these definatively. To do this, we need to think about how connect summing effects the determinant of a knot.

Here are two knots.

Denote the knot on the left K_1 and the knot on the right K_2.



From K_1 we obtain the matrix:

$$\begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

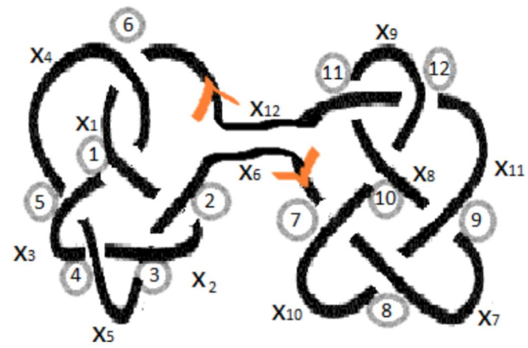
From K_2 we obtain the matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 2 & 0 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & -1 \end{pmatrix}$$

Recall the determinant of the knot is found by deleting any row and column of this matrix and then calculating the determinant.

Now consider $K_1\#K_2$, the connected sum of K_1 and K_2. From it, we obtain the matrix:

$$\left(\begin{array}{cccccc|cccccc}
2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & 0 & \boxed{0} & 0 & 0 & 0 & 0 & 0 & \boxed{-1} \\
\hline
0 & 0 & 0 & 0 & 0 & \circled{-1} & -1 & 0 & 0 & 2 & 0 & \circled{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1
\end{array}\right)$$



The circled elements highlight how this matrix has been changed from those above. The determinant of $K_1\#K_2$ is found by deleting any row and column of this matrix and then calculating the determinant. Convince yourself that the following proposition is true.

Proposition: $\det(K_1\#K_2)=\det(K_1)\det(K_2)$.

Problem: Use the above proposition to answer the following:
If both K_1 and K_2 are p colorable, then is the connected sum $K_1\#K_2$ p colorable?

If only one of K_1 and K_2 are p colorable, then is the connected sum $K_1\#K_2$ p colorable?

If neither of K_1 and K_2 are p colorable, then can the connected sum $K_1\#K_2$ be p colorable?

Note: Every knot K can be written as a connected sum of the unknot U and itself, that is, K=K#U. We call a knot K composite if we can find two knots K_1 and K_2 such that $K=K_1\#K_2$ where neither K_1 or K_2 is the unknot.

Sometimes it is obvious when a knot is composite, but other times it takes manipulating the diagram quite a bit to see that it is a connected sum of two other knots. For example, these are all composite knots: