Problem Solving Session: Invariants and Monoinvariants
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Big idea: An *invariant* is something that stays the same. A *monoinvariant* is something that only changes in one direction (eg. decreases, etc.). They are very useful tools for making your intuition rigorous.

Strategy: Eat pancakes! Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Discuss. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont give up after five minutes. Come back to it later. Ask.

1. I write 1, 2, 3, ..., 2017 on the blackboard. Then I play a game where I erase any two of numbers a and b, then write $a - b$. I keep doing this until there is only one number left. Is it possible to end up with 0?

2. Consider an 8×8 chessboard. Can you cover it with 2×1 dominoes? What if you remove a corner square of the chessboard, can you still cover it? What if you remove two corners? Does it matter if the corners are on the same side of the board or opposite sides of a diagonal?

3. Two players take turns breaking up an $m \times n$ chocolate bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into pieces along the subdivisions between its squares. The player who makes the last break wins. Does one of the players have a winning strategy?

4. Given n red points and n blue points in the plane, show that we can draw n non-intersecting line segments, each having one red endpoint and one blue endpoint.

5. On an $n \times n$ board there are n^2 squares, $n - 1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

6. Some people are in a building with several rooms. Each minute a person leaves a room and moves to another that has at least as many people. Show that eventually all of the people are in a single room.

7. Some number of frogs are squatting on a row of several lily pads in a swamp. Each minute, if there are two frogs on the same lily pad, and this pad is not at either end of the row, the two frogs may jump to two adjacent lily pads (in opposite directions). Prove that this process cannot be repeated forever.

8. A deck contains cards numbered $1, 2, ..., n$ in random order. Repeat: when the card on the top of the deck is numbered k, reverse the order of the first k cards. Show that 1 is eventually on the top of the deck.