Fundamental Group

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1. Show that composition of paths satisfies the following cancellation property:
   If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq f_0$, then $g_1 \simeq f_1$.

2. Show that the following three conditions are equivalent:
   a. Every continuous loop $f : S^1 \to X$ is homotopic to a constant loop.
   b. Every continuous $f : S^1 \to X$ extends to a continuous $g : D^2 \to X$. That is, g restricted to the boundary of the disk should agree with the map f.
   c. $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Bonus Problems:

3. A corollary of Borsuk-Ulam states that if you cover the sphere S^2 with three closed sets A_1, A_2, A_3, then one of the sets contains a pair of antipodal points. To prove this, introduce the functions $d_i : S^2 \to \mathbb{R}$ defined by $d_i(x)$ the minimum distance from x to some point of A_i. (Hence, $d_i(x) = 0$ if and only if $x \in A_i$.) Now consider the function $D : S^2 \to \mathbb{R}^2$ defined by $D(x) = (d_1(x), d_2(x))$. Complete the details of the proof.

   Note: Similarly, if you cover the circle S^1 with two closed sets, then one of those sets will contain a pair of antipodal points. (Try it!) You can extend this result to higher dimensions, too.

4. Can you describe the fundamental group of the torus?